Catalytic Decoupling

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Structure

Introduction

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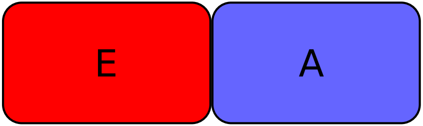

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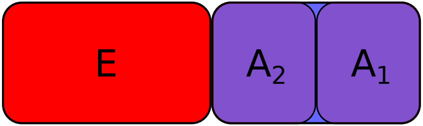

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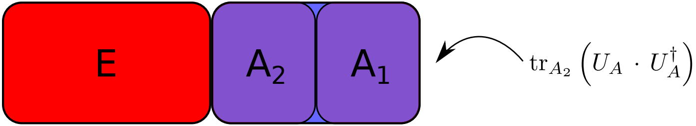

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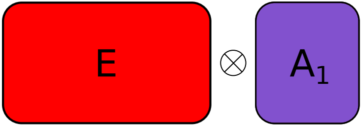

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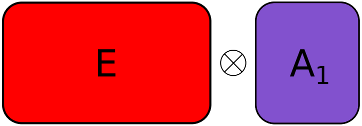
- how big do we have to choose A_2?



Applications:

Applications:

- Quantum state merging (Horodecki, Oppenheim, Winter), "mother protocol" (Abeyesinghe et al.)
- Quantum state redistribution (Devetak, Yard)
- Channel coding
- Quantum thermodynamics
- Solid state physics
- Black hole radiation

...

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- $I^{\varepsilon}_{\mathrm{max}}(A : B)_\rho = \mathrm{min}_{\tilde{\rho}}\, I_{\mathrm{max}}(A : B)_{\tilde{\rho}}$

One shot secoupling

Theorem (General one shot decoupling; Dupuis, Berta, Wullschleger, Renner (DBWR))

Let $\varepsilon > 0$*,* ρ_{AE} *a bipartite quantum state, and let* $\mathcal{T}_{A\to B}$ *be a quantum channel with Choi-Jamiołkowski representation* $\tau_{AB} = J(\mathcal{T})$ *such that* $tr(\tau_{AB}) \leqslant 1$*. Then, we have that*

$$\int_{\mathrm{U}(A)} \left\| \mathcal{T}_{A\to B}(U_A \rho_{AE} U_A^\dagger) - \tau_B \otimes \rho_E \right\|_1 dU \\ \leq 2^{-\frac{1}{2} H_{\min}^{\varepsilon}(A|E)_\rho - \frac{1}{2} H_{\min}^{\varepsilon}(A|B)_\tau} + 12\varepsilon \ ,$$

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Then, we have

$$\int_{\mathrm{U}(A)} P\left(tr_{A_2}\left(U_A \rho_{AE} U_A^\dagger\right), \frac{1_{A_1}}{|A_1|} \otimes \rho_E\right) \mathrm{d}\, U_A \leq \mathcal{O}(\varepsilon).$$

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$\approx$Theorem

Let $\rho_{AE} = {\rho'_{A'B'}}^{\otimes n}$ be a bipartite i.i.d. quantum state, and let $\mathcal{H}_A \cong \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ such that

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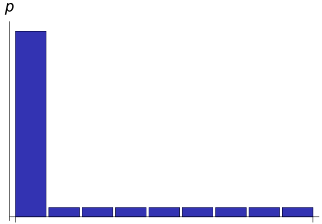

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$\Rightarrow$ $H^{\varepsilon}_{\max}(A)_\rho - H^{\varepsilon}_{\min}(A|E)_\rho \approx \log n$,

while A should be considered decoupled from the trivial E

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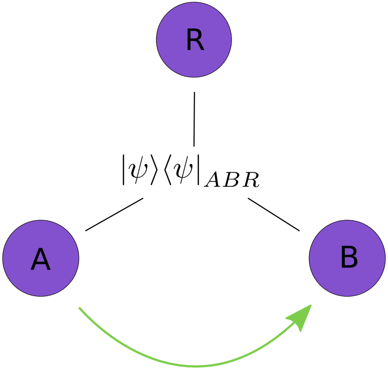

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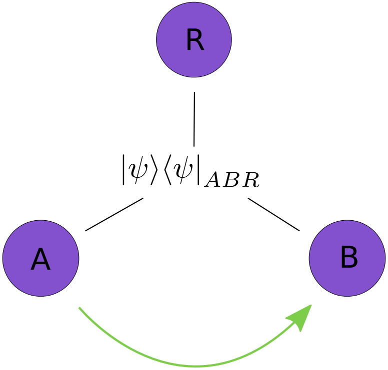

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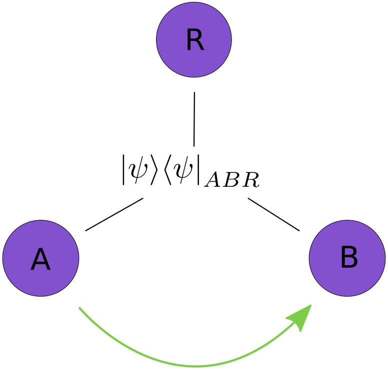

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... uses different technique called convex split lemma.

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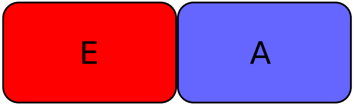

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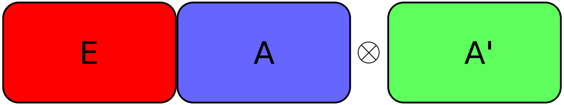

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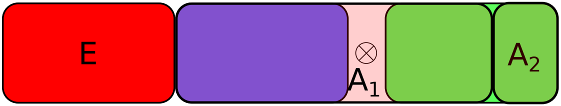

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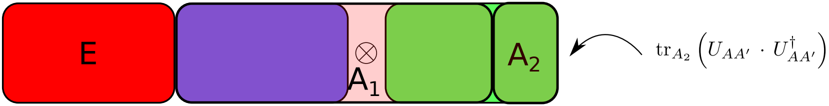

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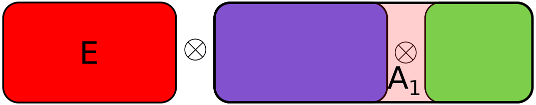
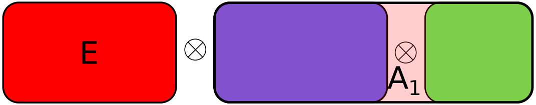


Catalytic decoupling

- Generalize decoupling twofold:
- drop randomization condition
- allow free mixed ancillary states

In pictures, step by step:

- Bipartite system AE
- add ancillary system A' in a fixed state
- divide system AA' into two parts, $AA' \cong A_1 \otimes A_2$
- apply a unitary
- trace out A_2
- how big do we have to choose A_2 here?



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- Two proofs, one using the techniques from Anshu et al. and Berta et al. respectively

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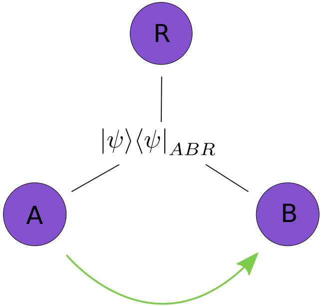

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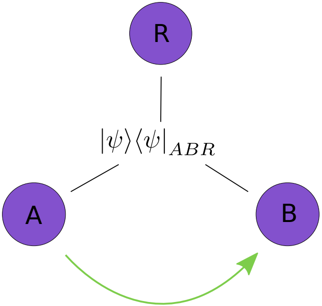

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⇒ One-shot local erasure of correlations can be done with $2^{I_{\max}^{\varepsilon}(E:A)}$ unitaries using an ancilla

Proofs

Proof ingredient

Lemma (Convex split lemma; Anshu, Devabathini, Jain)

Let $\rho \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_E)$ *and* $\sigma \in \mathcal{B}(\mathcal{H}_E)$ *be quantum states,* $k = D_{max}(\rho_{AE} \| \rho_A \otimes \sigma_E)$ *and* $0 < \delta < \frac{1}{6}$*. Define*

$$n = \begin{cases} 1 & k \leq 3\delta \\ \left\lceil \frac{8 \cdot 2^k \log\left(\frac{k}{\delta}\right)}{\delta^3} \right\rceil & \text{else} \end{cases} .$$

Then the state

$$\tau_{AE_1 \dots E_n} = \frac{1}{n} \sum_{j=1}^{n} \rho_{AE_j} \otimes \left(\sigma^{\otimes (n-1)}\right)_{E_{j^c}} \tag{1}$$

is almost product,

$$I(A; E_1 \dots E_n)_\tau \leq 3\delta \quad \text{as well as} \quad P\left(\tau_A \otimes \tau_{E_1 \dots E_n}, \tau_{AE_1 \dots E_n}\right) \leq \sqrt{6\delta}.$$

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- rotate standard basis of C to Bell basis
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1. Proof sketch (ctd.)

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- use half an embezzling state (van Dam and Hayden) to "unembezzle" maximally mixed states of different sizes

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- especially: finite block length regime?

Summary

- We defined catalytic decoupling
- We characterized it, unifying two techniques in one-shot quantum communication
- second order asymptotics for i.i.d.
- Applications: local erasure of correlations, decoupling proof of one-shot coherent state merging

As this is the last talk: A big "Thank you" to the organizers for the great workshop!

Questions?