Weak approximate unitary designs and applications to quantum encryption

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Results — overview

- We show that **very small approximate unitary t-designs** exist when the approximation error is measured in certain **non-stabilized norms**.
- This extends the line of work started by Hayden, Leung, Shor and Winter (CMP 04)
- Our proofs rely on a technical result by Aubrun (CMP 09)
- As an **application**, we exhibit a probabilistic construction of a **quantum encryption scheme** that is non-malleable against adversaries with no (or limited) quantum side information.

Introduction

The Haar measure

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- Can be pushed forward to homogeneous spaces
- Quantum information: Unitary group, projective space (=set of pure states)

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- Entanglement theory

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Twirling channels

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$$T^{(1,1)}(X) = \mathop{\mathbb{E}}_{U\sim\mathrm{Haar}_d}\left[\left(U\otimes\bar{U}\right)X\left(U^\dagger\otimes U^T\right)\right]$$

Approximate unitary designs

$D \subset H$ is a unitary t-design $\Leftrightarrow$ $$T^{(t)} = T^{(t)}_D := \frac{1}{|D|} \sum_{U \in D} U^{\otimes t}(\,\cdot\,)\left(U^{\otimes t}\right)^\dagger$$

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In quantum information theory:
$\|\cdot\|_\diamond$ most natural

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Plenty of constructions: Random quantum circuits (BHH 16), random quantum circuits with a lot of structure (HMMHEGR 20, also CLLW 15, NHMW 17 for 2-designs)

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Pauli Group

Standard example of a 1-design:

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Maximally entangled state

D_n has minimal size: $T_{D_n}^{(1)} \otimes \mathrm{id}(|\phi^+\rangle\langle\phi^+|) = \mathbb{1},$

$$|D_n| \geq \mathrm{rk}\left(T_{D_n}^{(1)} \otimes \mathrm{id}(|\phi^+\rangle\langle\phi^+|)\right) = 2^{2n}$$

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Naively, 2^n elements could be enough!

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No side information: 1-to-1-norm $\|\Phi\|_{1\to 1} = \sup_X \frac{\|\Phi(X)\|_1}{\|X\|_1}$

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$\exists D : |D| \leq \mathcal{O}(n 2^n \varepsilon^{-2})$ s.t.

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Independent Haar-random unitaries work whp.!

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For D with $|D| = \Omega(n^6 2^n \varepsilon^{-2})$ independently random 1-design elements,

$$2^n \left\| T^{(1)} - T_D^{(1)} \right\|_{1\to\infty} \leq \varepsilon$$

with constant probability.

Results

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For $t = 2$: Variants for $T^{(1,1)}$ and T^{ch}:

$$T^{ch}(\Phi)(X) = \mathop{\mathbb{E}}_{U\sim \mathrm{Haar}_d} \left[U^\dagger \Phi(UXU^\dagger)U\right]$$

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T^{ch} is a "super-duper-operator" 😉. Define $\| \cdot \|_{\diamond\to\diamond}$

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$\boxed{\begin{array}{l}\text{Definition:} \\ \text{An } n \text{ qubit weak } \varepsilon\text{-approximate unitary } t\text{-design is a finite} \\ \text{subset } D \subset U(2^n) \text{ such that} \\ \qquad\qquad \left\| T^{(t)} - T^{(t)}_D \right\|_{1\to 1} \leq \varepsilon \end{array}}$

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Can we generalize the result by Aubrun?

Let's look inside (Aubrun 09)….

Key lemma in Aubrun 09:

Lemma 5. *Let* $U_1, \dots, U_N \in \mathcal{U}(d)$ *be* deterministic *unitary operators and let* (ε_i) *be a sequence of independent Bernoulli random variables. Then*

$$\mathbf{E}_\varepsilon \sup_{\rho \in \mathcal{D}(\mathbf{C}^d)} \left\| \sum_{i=1}^N \varepsilon_i U_i \rho U_i^\dagger \right\|_\infty \leqslant C (\log d)^{5/2} \sqrt{\log N} \sup_{\rho \in \mathcal{D}(\mathbf{C}^d)} \left\| \sum_{i=1}^N U_i \rho U_i^\dagger \right\|_\infty^{1/2}$$

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Banach space geometry!

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Proof sketch of subsampling weak 1-design given this Lemma:

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Conclude that weak 1-design error is bounded if $N \geq \mathrm{polylog}(d) d^2 \|T^{(1)}(\rho)\|_\infty$

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Observation: Yields interesting bound whenever subsampling from a design approximating a channel with small $1 \to \infty$ norm!!!

Abstracting the technique of (Aubrun 09)

Lemma (relatively straightforward from (Aubrun 09)):

Let $\hat{D} \subset U(d)$ be such that

$$\left\| T_{\hat{D}}^{(1)} \right\|_{1\to\infty} \leq \delta.$$

Then the subsampling design D fulfils

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Can apply this to t-design setting! Take full t-design $\tilde{D} \subset U(d)$, set

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$\Rightarrow$ need bound on $\left\| T^{(t)} \right\|_{1 \to \infty}$.

Abstracting the technique of (Aubrun 09)

$\boxed{\begin{array}{l}\text{Lemma (relatively straightforward from (Aubrun 09)):} \\ \text{Let } \hat{D} \subset U(d) \text{ be such that} \\ \qquad \left\| T^{(1)}_{\hat{D}} \right\|_{1\to\infty} \leq \delta. \\ \text{Then the subsampling design } D \text{ fulfils} \\ \qquad d \left\| T^{(1)}_{D} - T^{(1)}_{\hat{D}} \right\|_{1\to\infty} \leq \varepsilon \\ \text{provided that } |D| \geq \mathrm{polylog}(d)\dfrac{\delta d^2}{\varepsilon^2}\end{array}}$

Can apply this to t-design setting! Take full t-design $\tilde{D} \subset U(d)$, set

$$\hat{D} = \{U^{\otimes t} \,|\, U \in \tilde{D}\} \subset U(d^t)$$

$\Rightarrow$ need bound on $\left\| T^{(t)} \right\|_{1\to\infty}$.

Representation theory!!!

Representation theory of $U(d)$

$U(d) \to U(d^t)$, $U \mapsto U^{\otimes t}$ is a representation of $U(d)$

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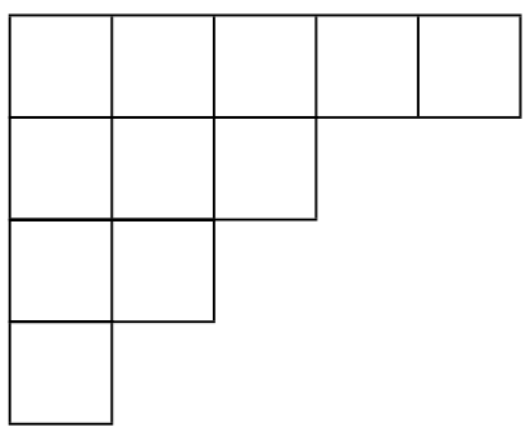
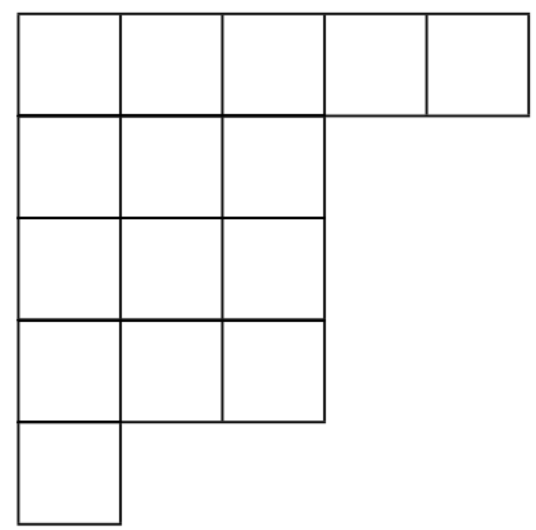
Irreducible representations of $U(d)$: $V_{\lambda,d}$ labeled by *Young diagrams*

Examples:



$\lambda = (5,3,2,1) \vdash 11$



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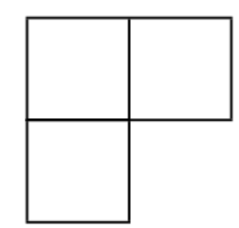

$\lambda = (2,1) \vdash 3$

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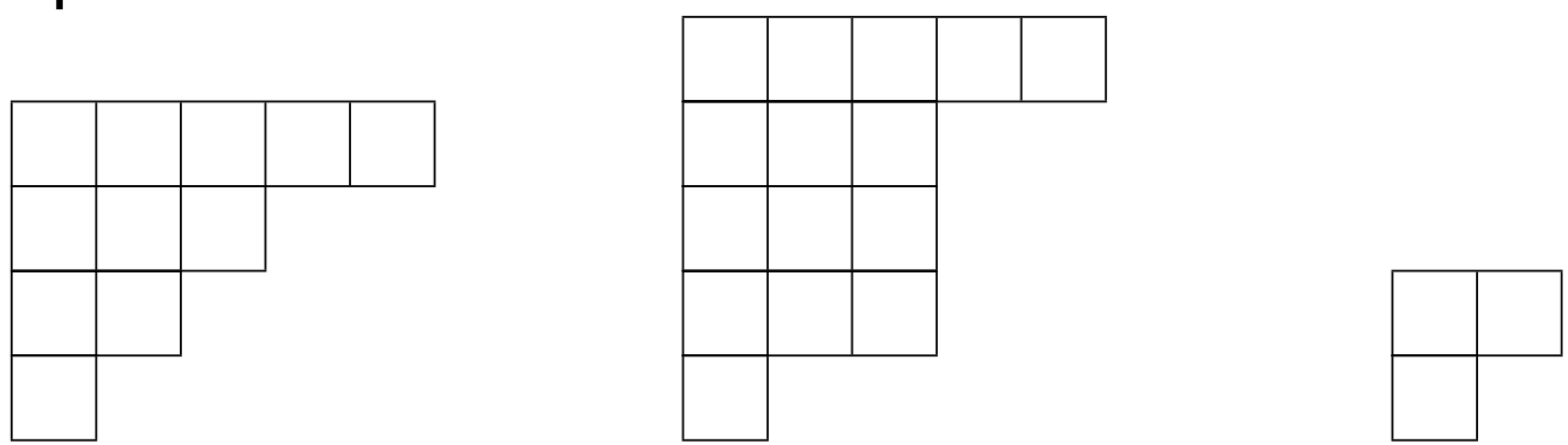

$\lambda = (5,3,2,1) \vdash 11 \quad \lambda = (5,3^3,1) \vdash 15 \quad \lambda = (2,1) \vdash 3$

Dimension of $V_{\lambda,d}$: combinatorial formula in λ and d

Analyzing $T^{(t)}$

Decompose $\left(\mathbb{C}^d\right)^{\otimes t}$ into irreps of $U(d)$:

$$\left(\mathbb{C}^d\right)^{\otimes t} \cong \bigoplus_{\lambda \vdash_d t} V_{\lambda,d} \otimes \mathbb{C}^{m_\lambda}$$

Young diagrams with t boxes
(and at most d rows)

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Schur's Lemma implies

$$T^{(t)}(X) = \bigoplus_{\lambda \vdash_d t} \tau_{V_{\lambda,d}} \otimes \mathrm{Tr}_{V_{\lambda,d}}\left[P_\lambda X\right]$$

Maximally mixed state

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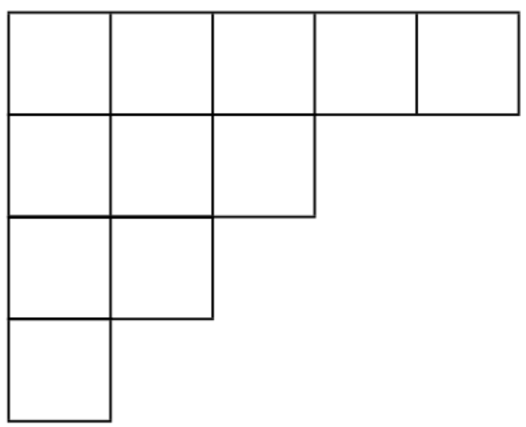

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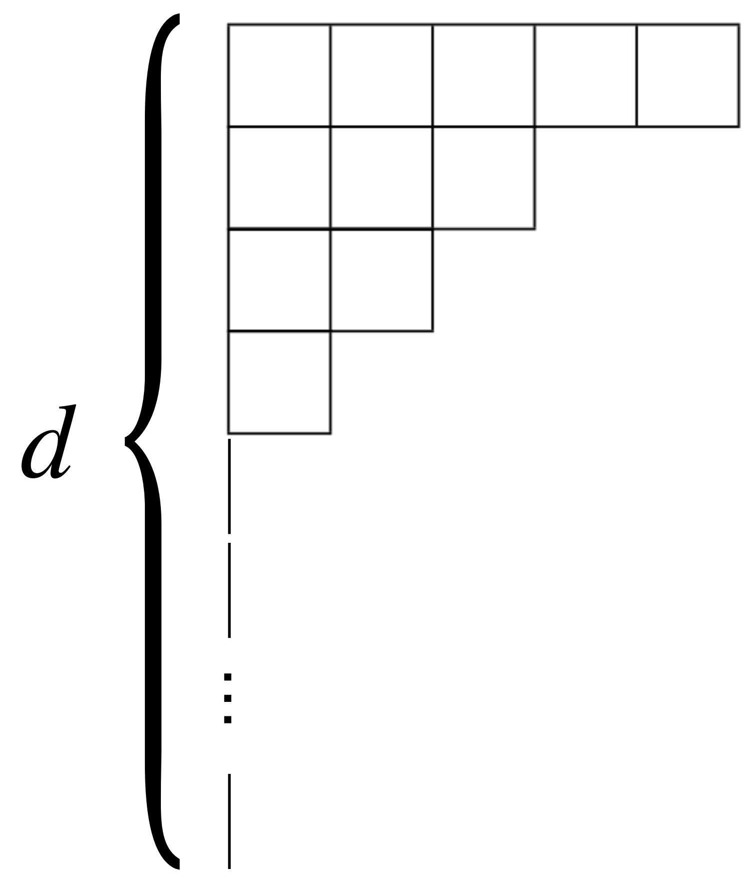

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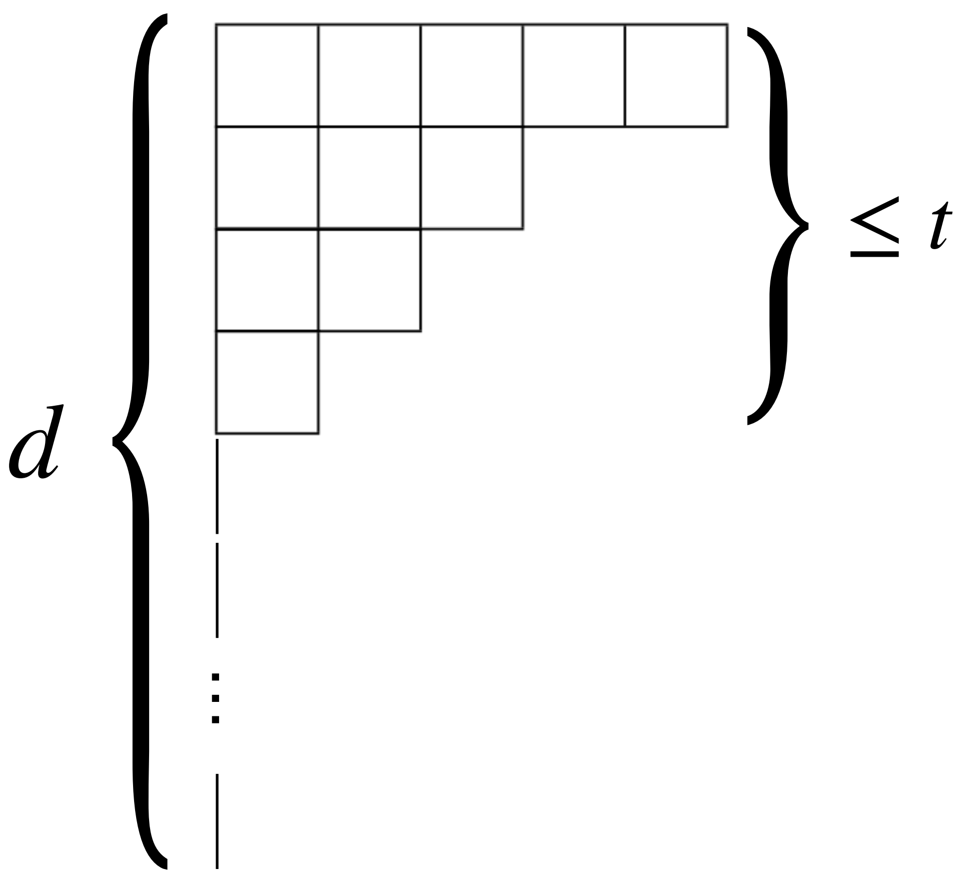

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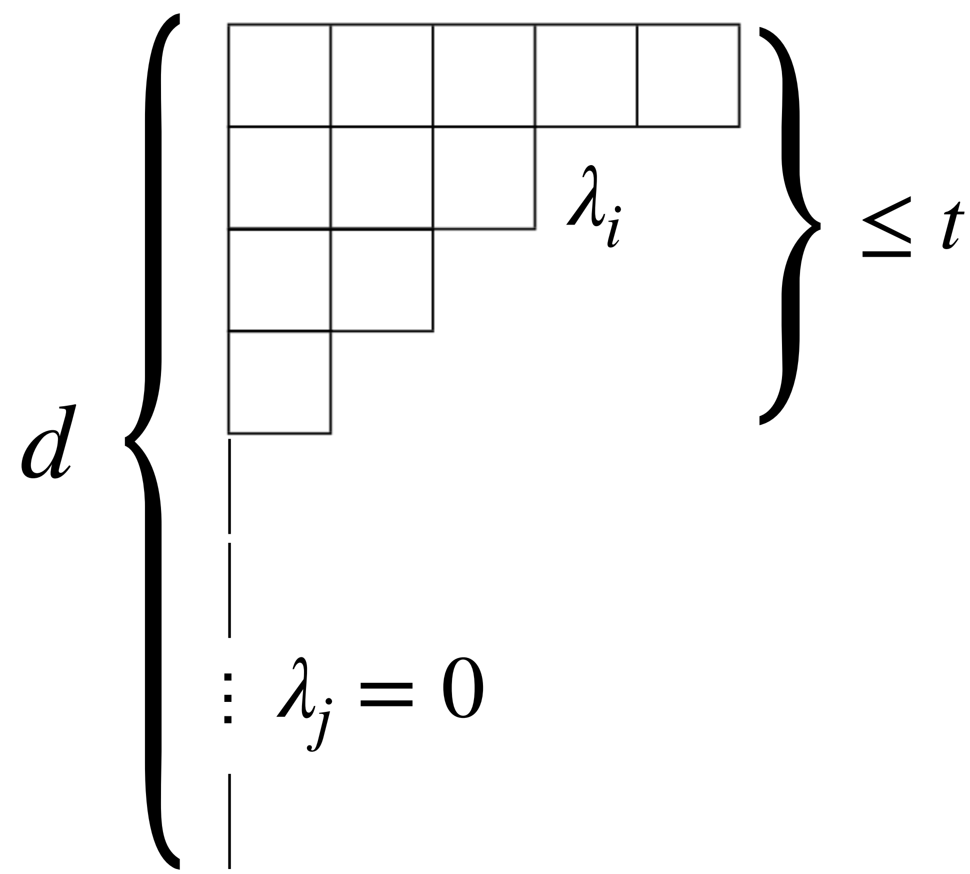

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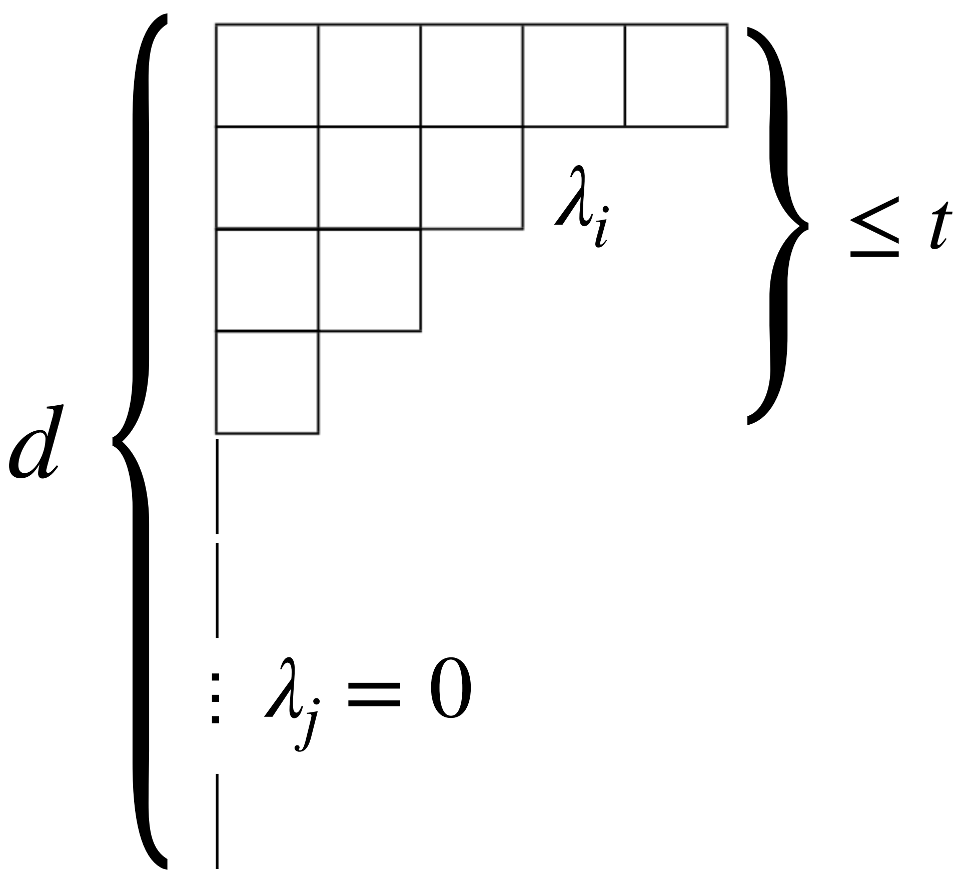
"Telescopic" product



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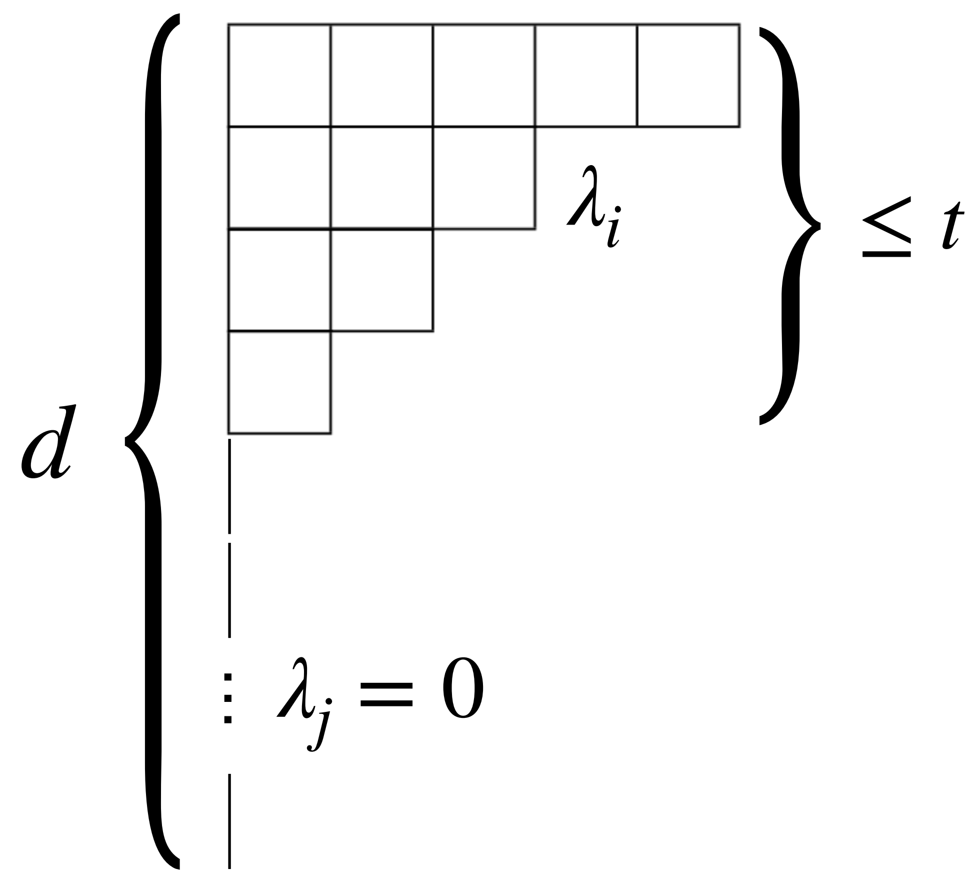

$\Rightarrow \dim V_{\lambda,d} \geq \left(\frac{d}{2t}\right)^t$ for $\lambda \vdash t$

Main result

Theorem (Lancien, CM):

Let $\hat{D} \subset U(d)$ be a unitary t-design. Then the subsampling design D fulfils

$$d^t \left\| T_D^{(t)} - T^{(t)} \right\|_{1\to\infty} \leq \varepsilon$$

provided that $|D| \geq \mathrm{poly}(\log d, t)\varepsilon^{-2}(td)^t$

Variant for $T^{(1,1)}$

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$|\phi^+\rangle\langle\phi^+|$, invariant

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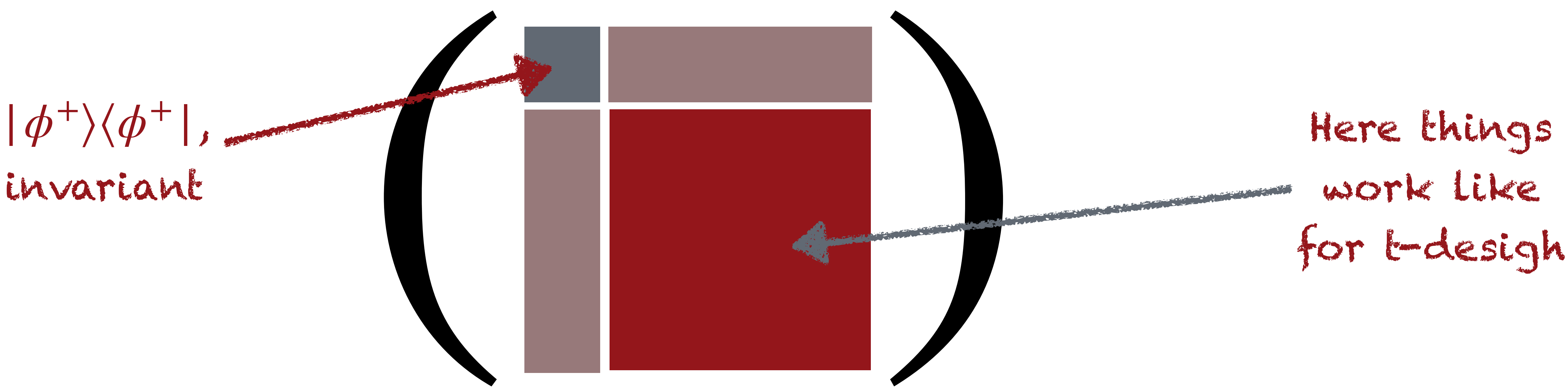

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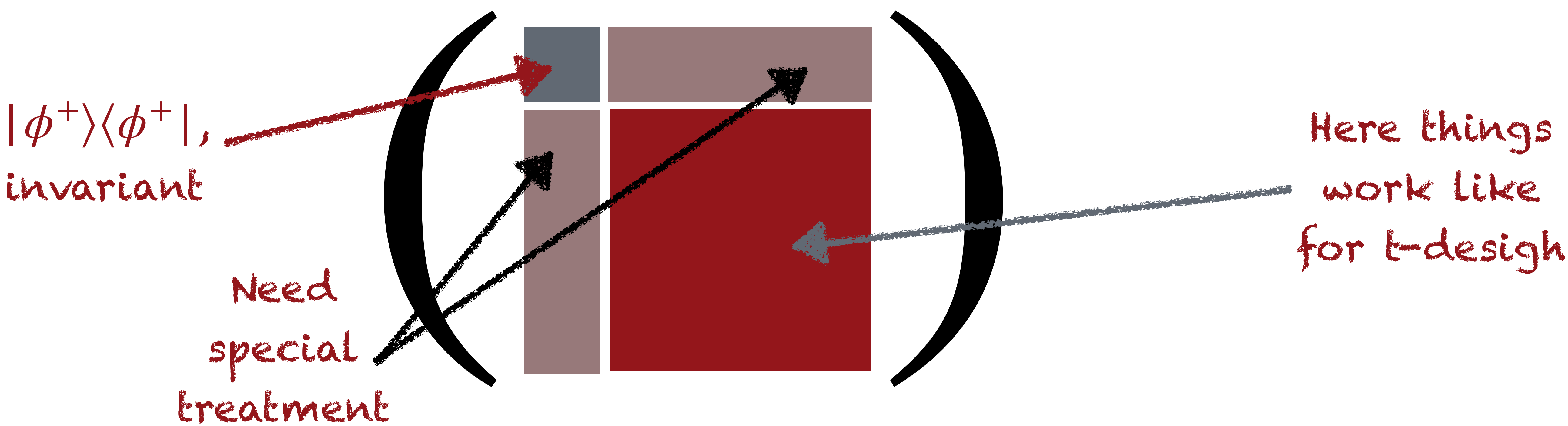
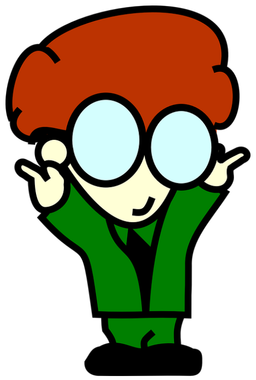


Application: Non-malleable encryption

Non-malleability



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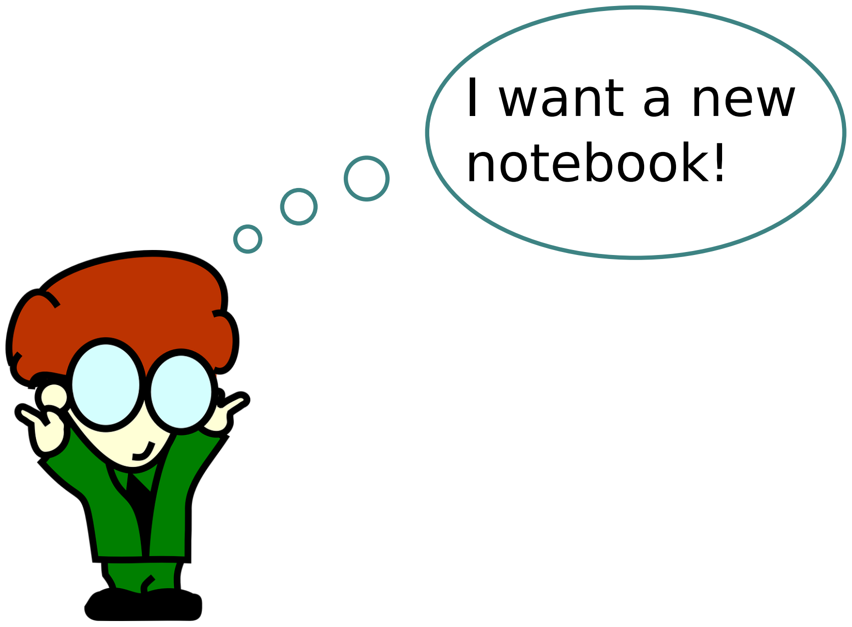

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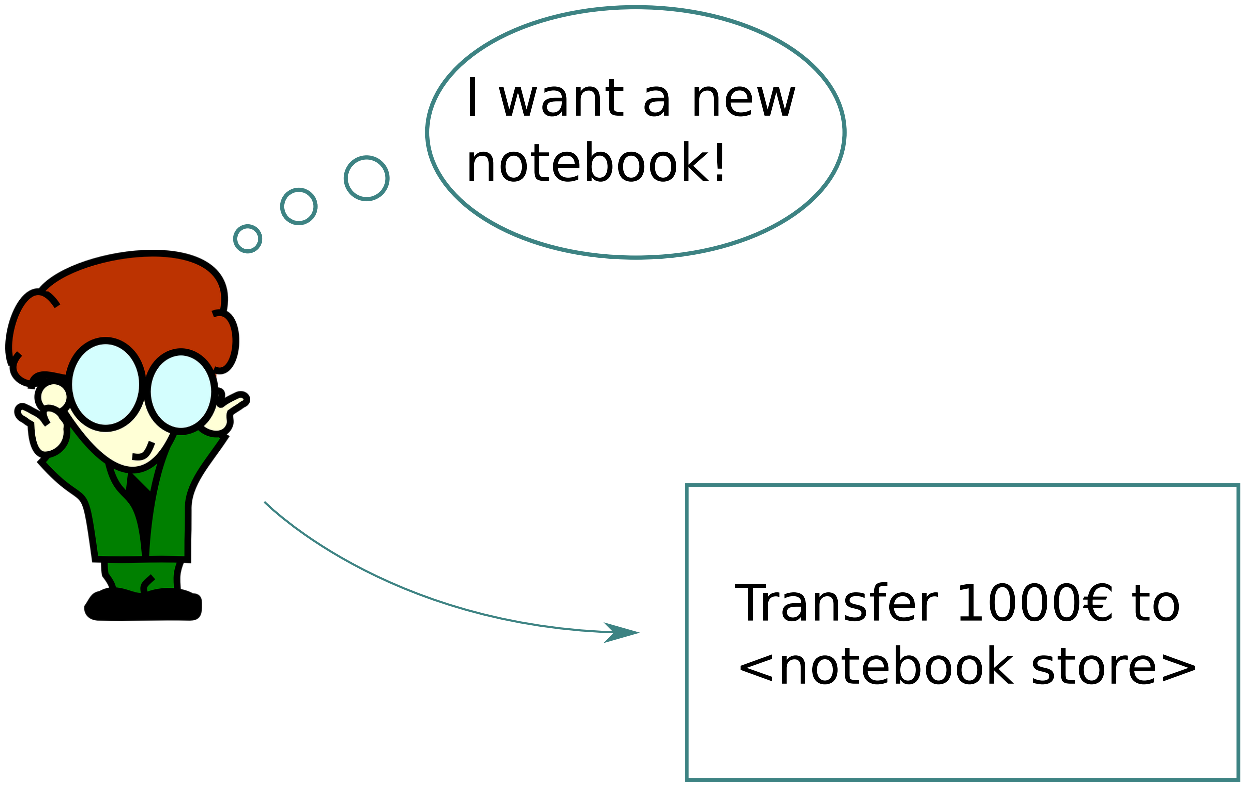

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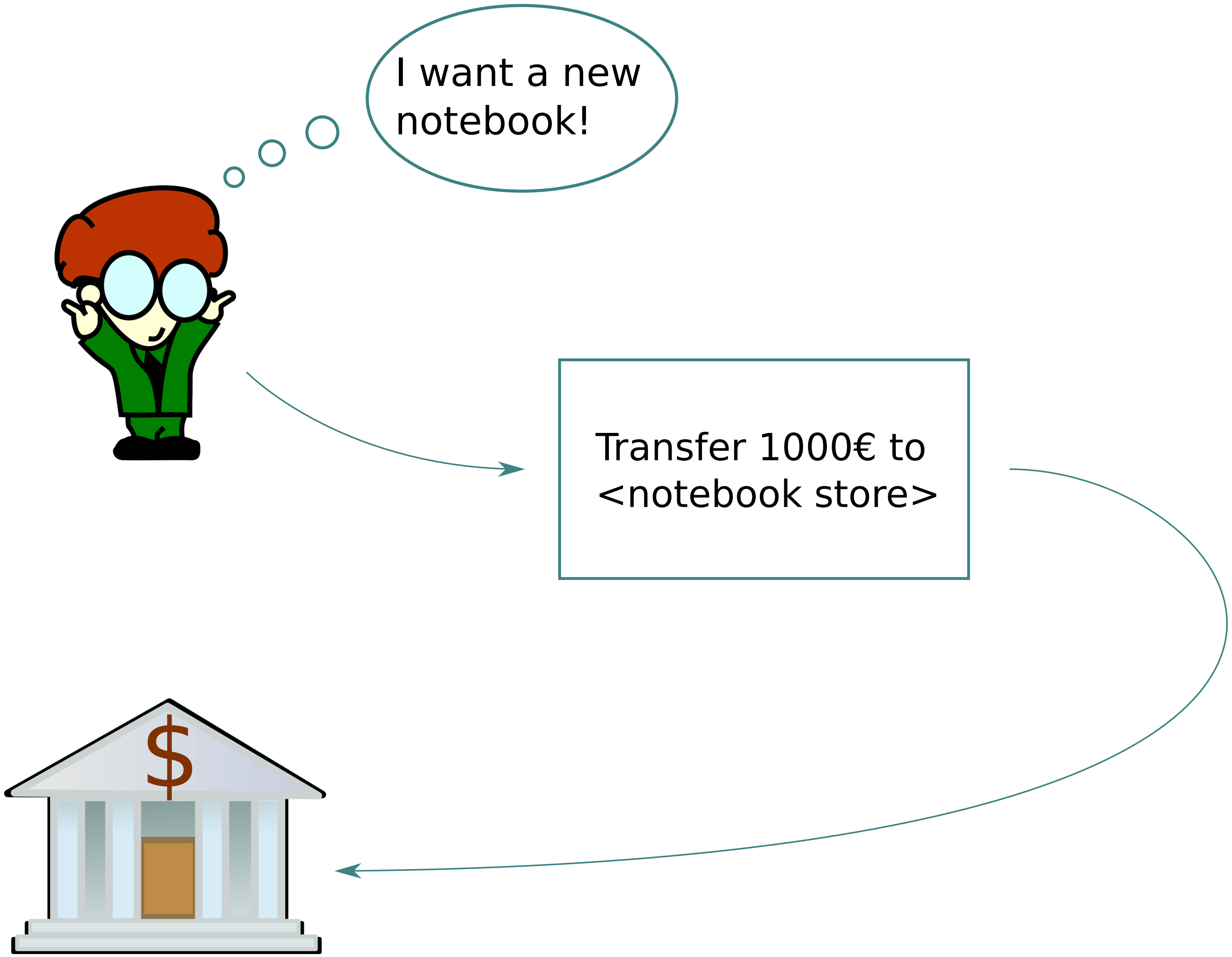

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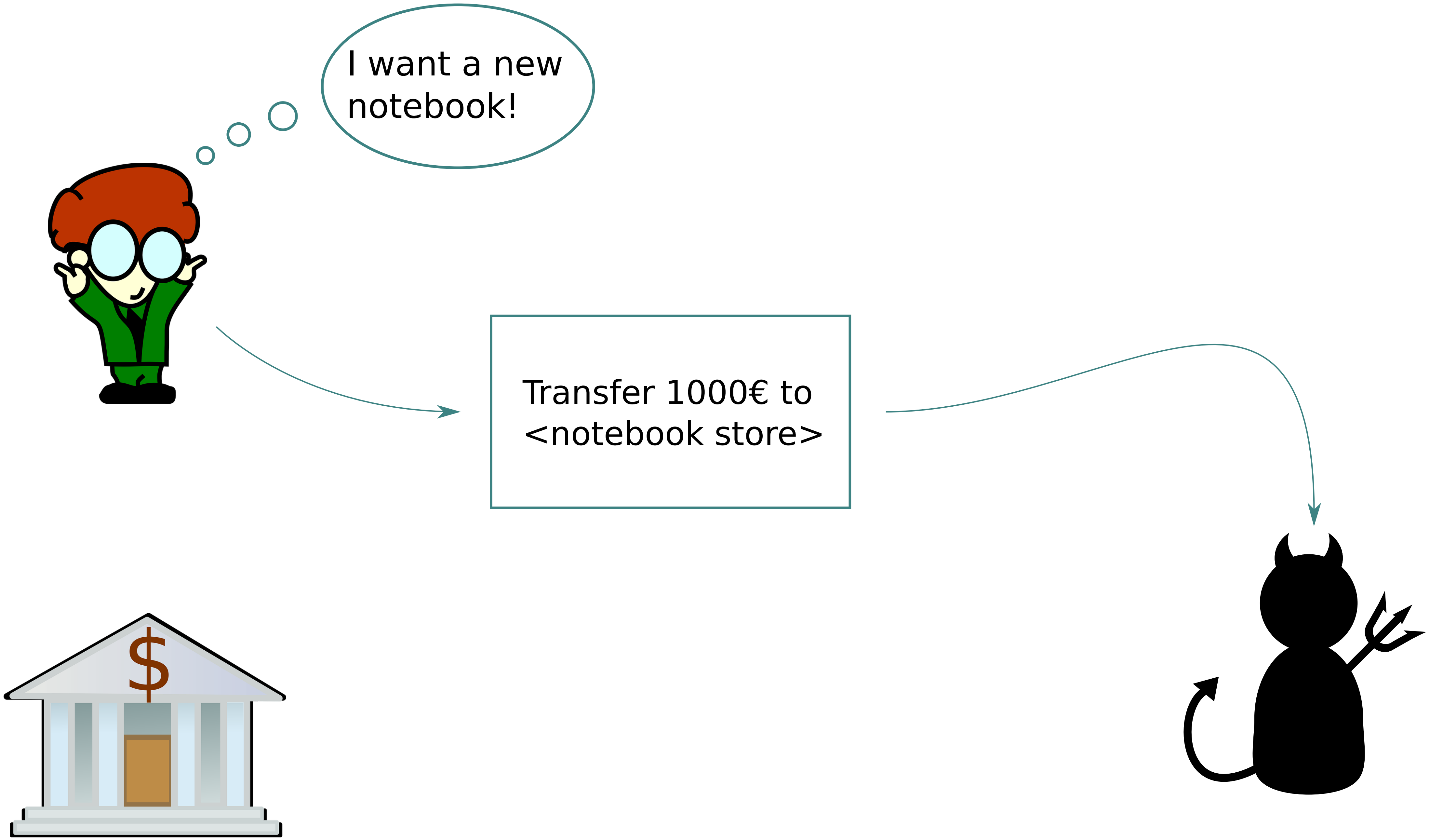

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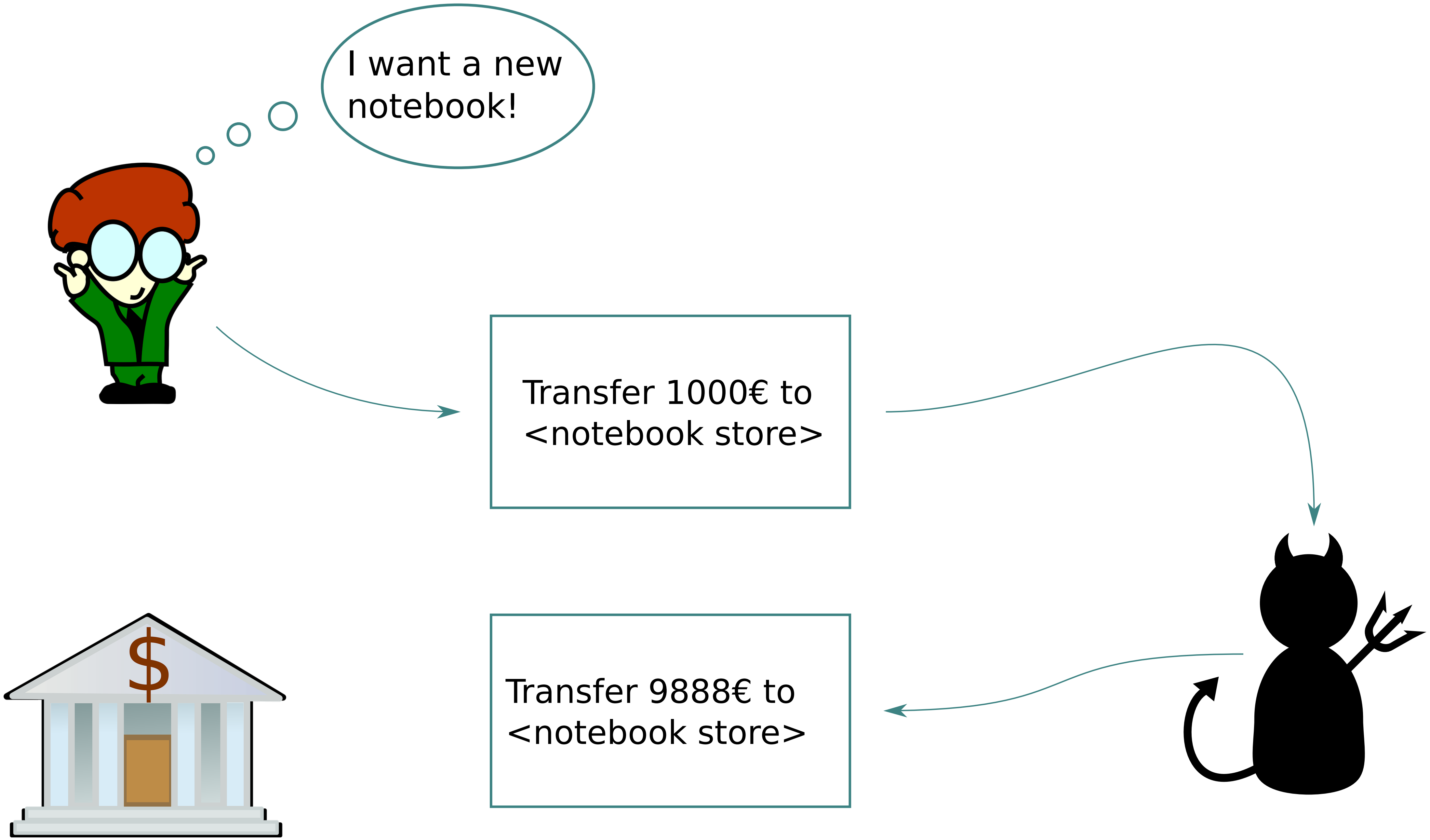

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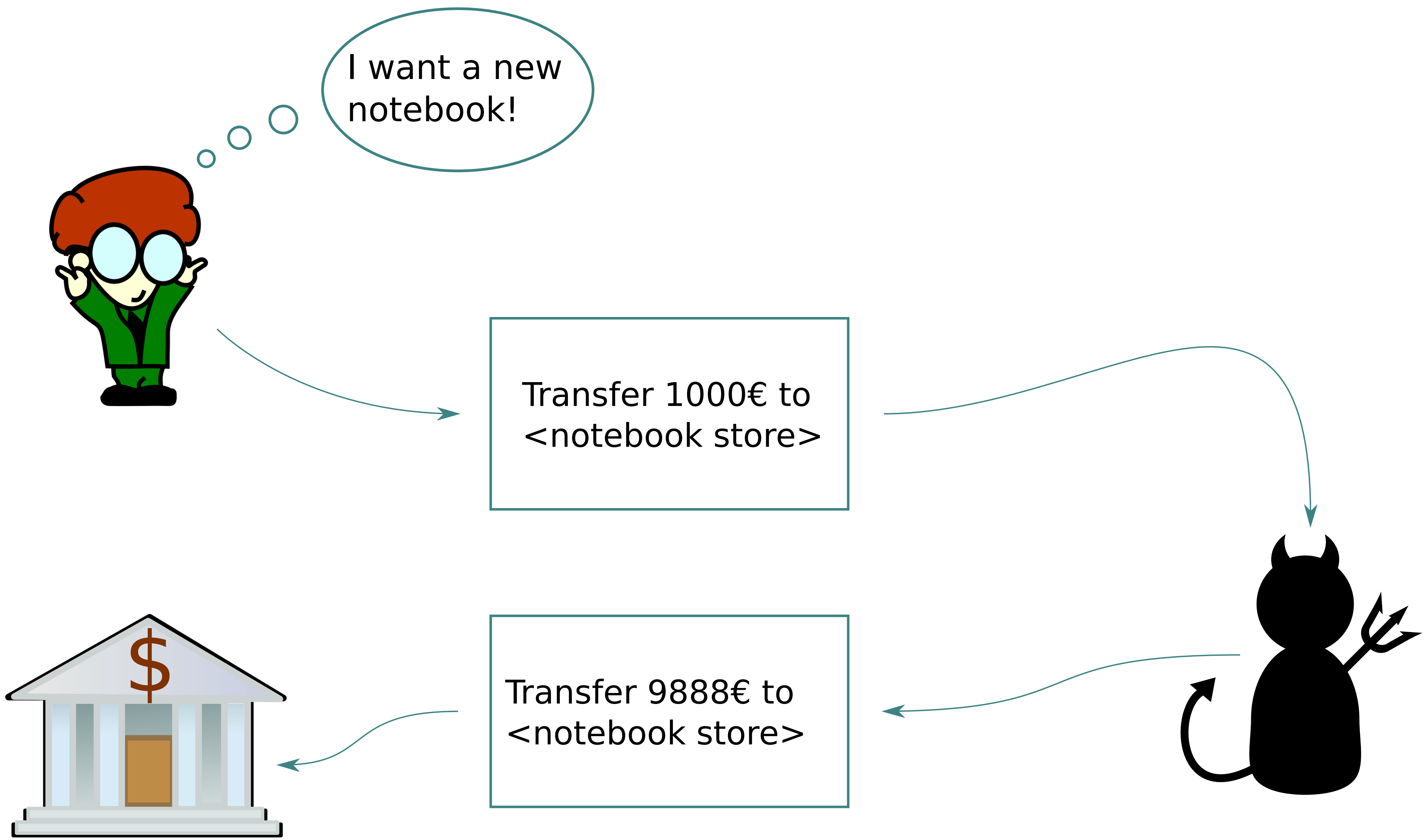

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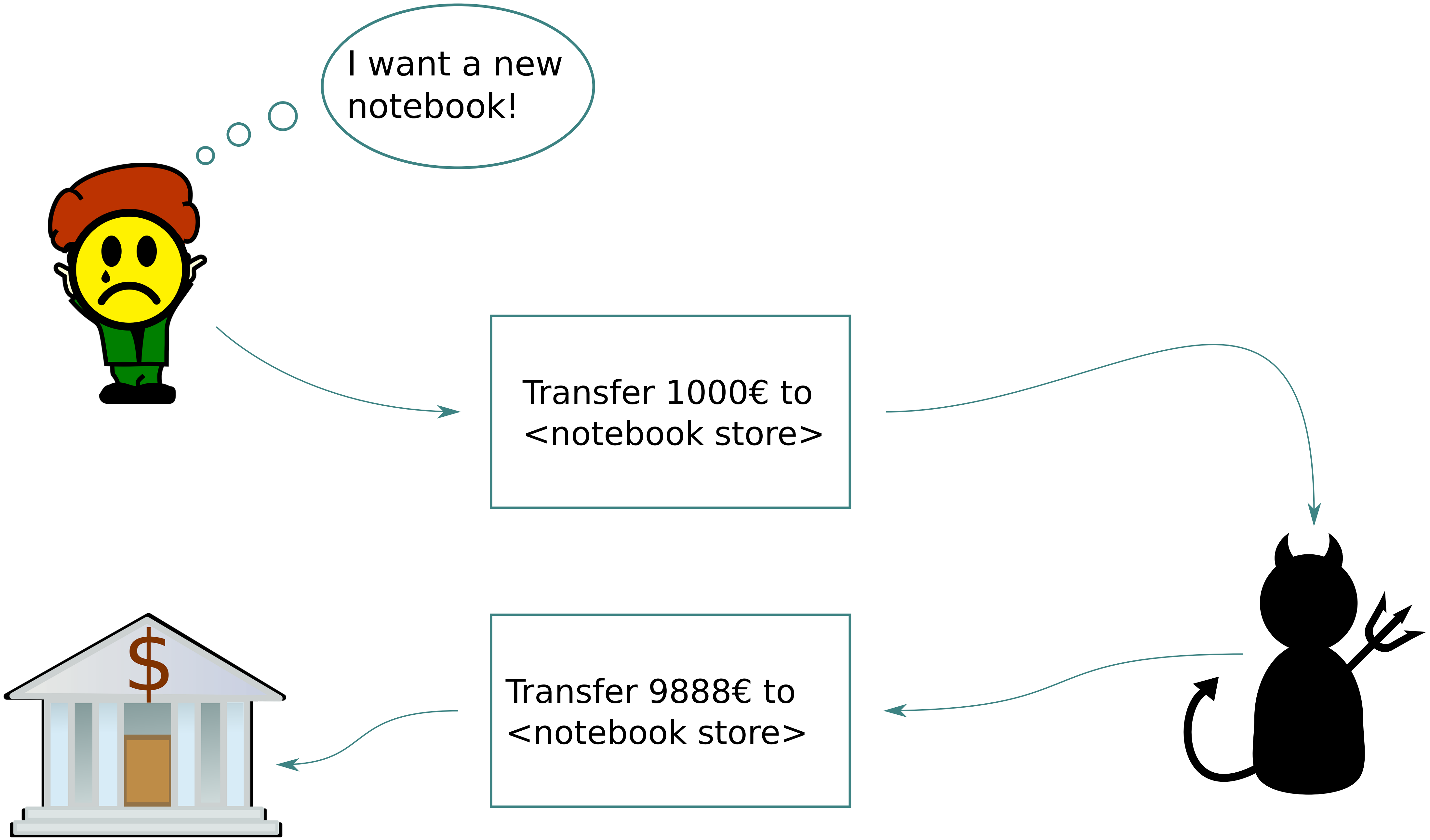

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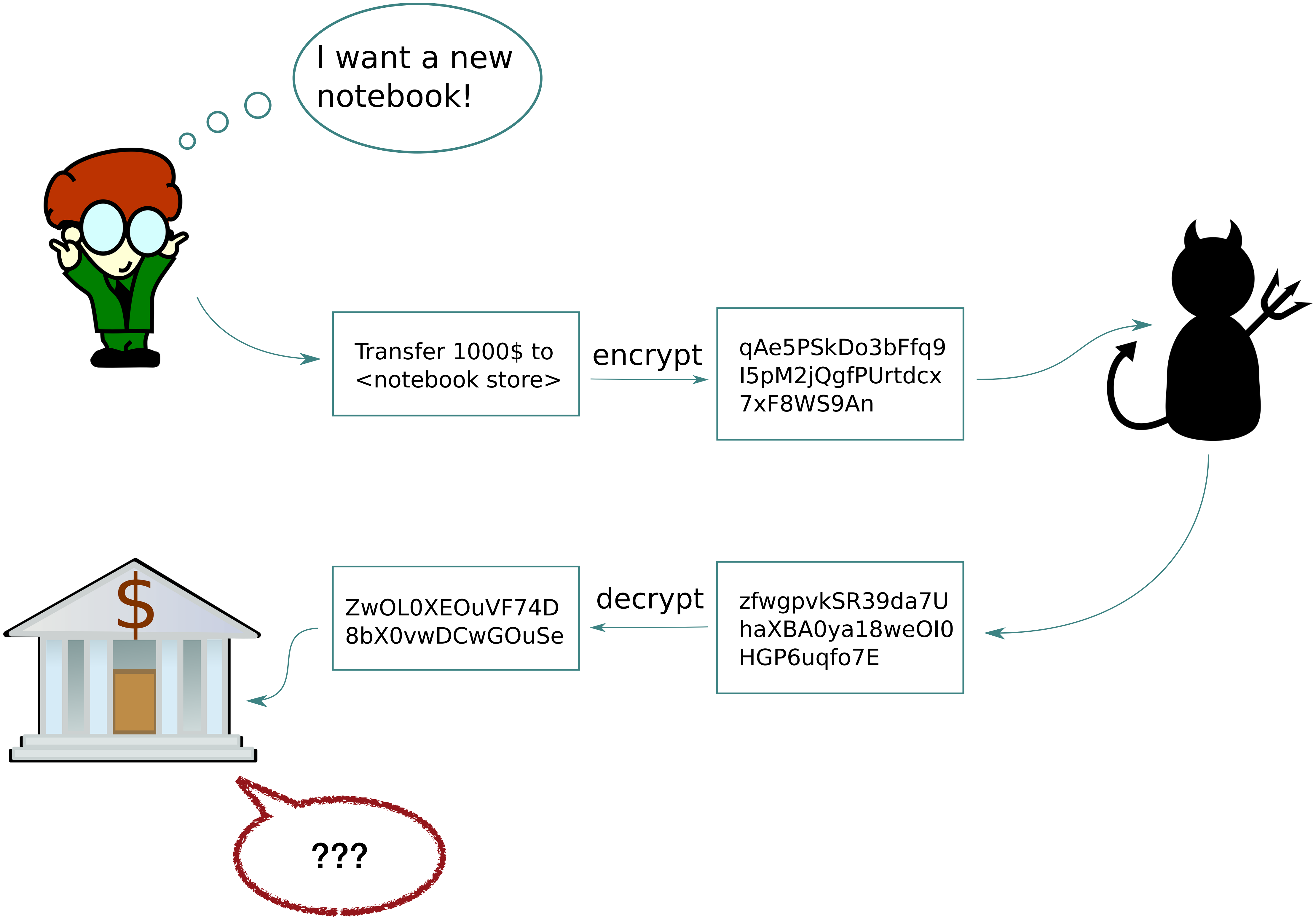

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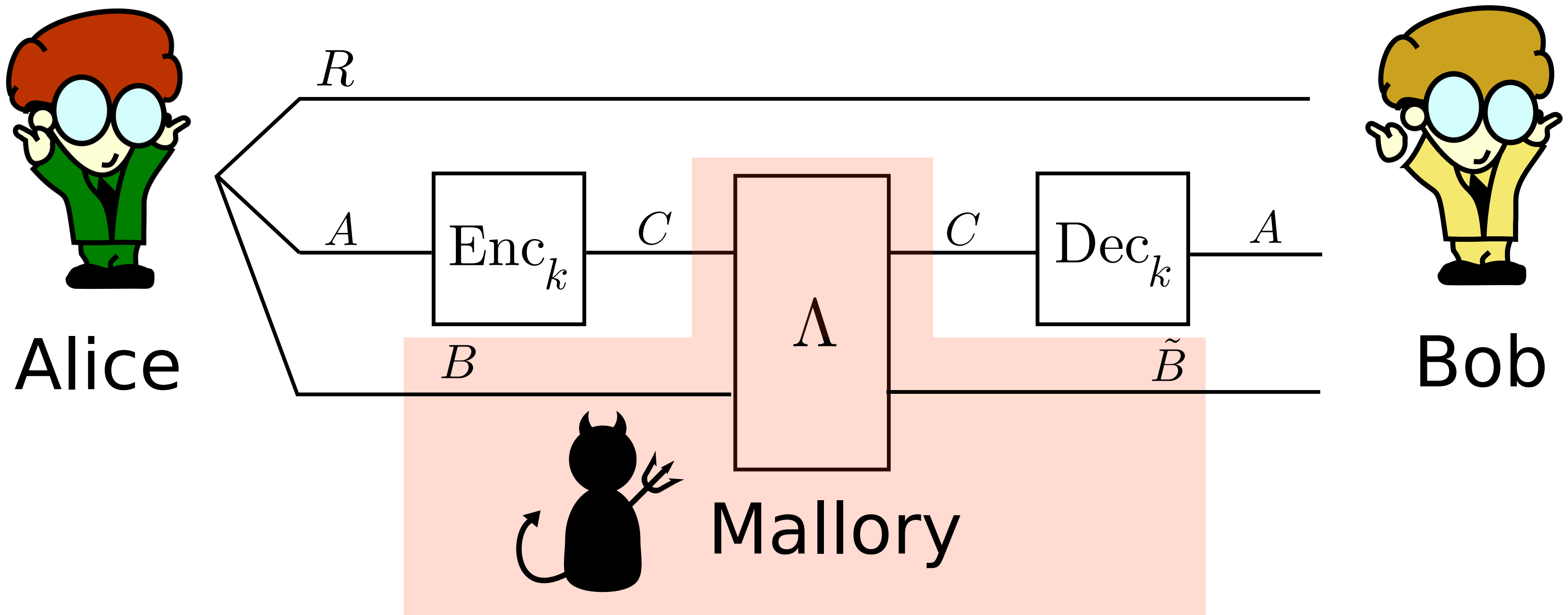

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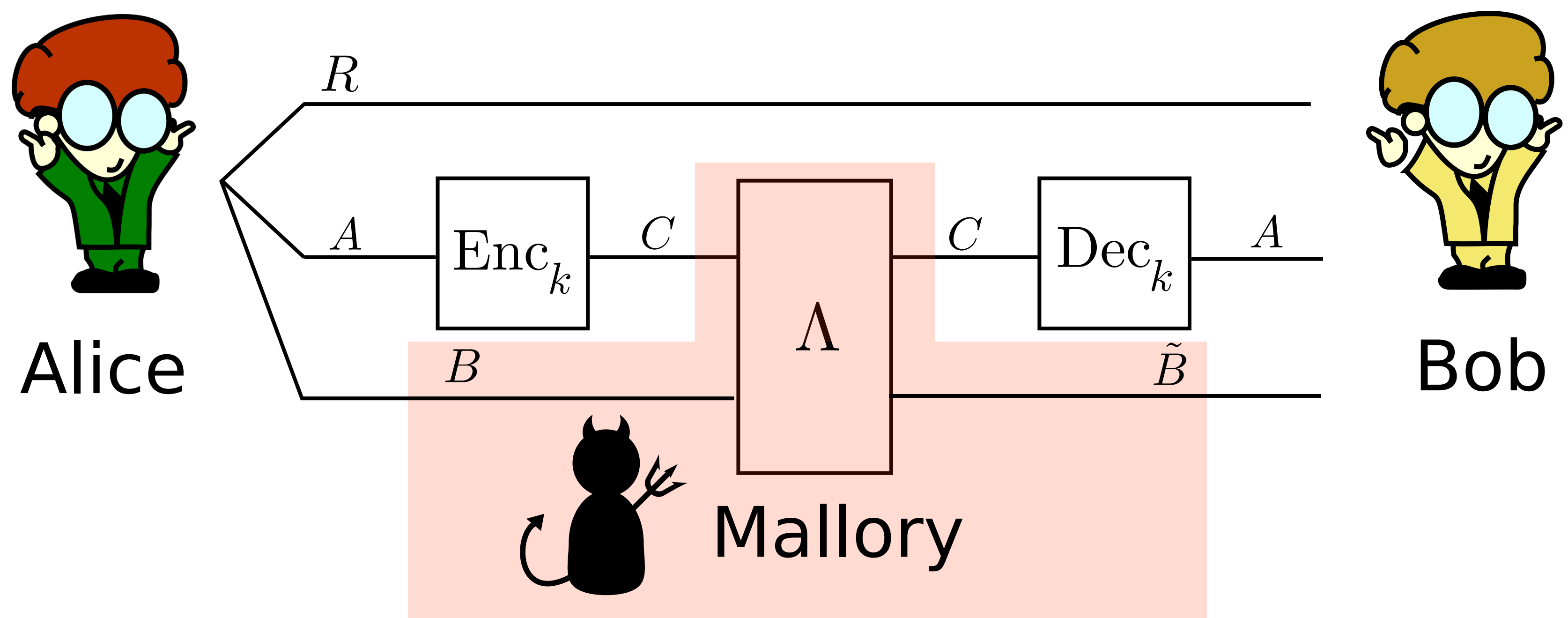

Encryption with 2-design works (ABW 09; AM 17). Key length: $\sim 4n$ bits for n qubit encryption (shorter keys and larger ciphertexts are also possible, BCGST04)

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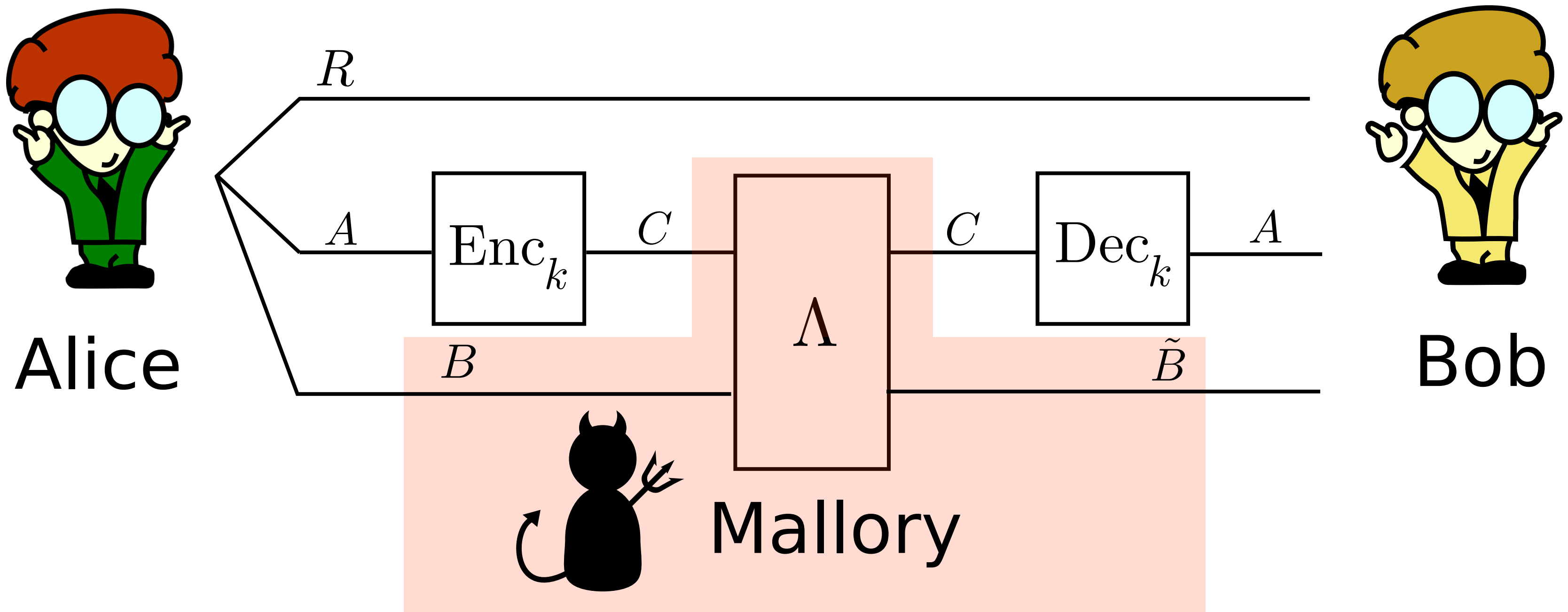

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Weak designs: randomized construction of unitary encryption scheme nm against adversaries with $\leq s$ bits of quantum memory, key length $\sim 2(n+s)$

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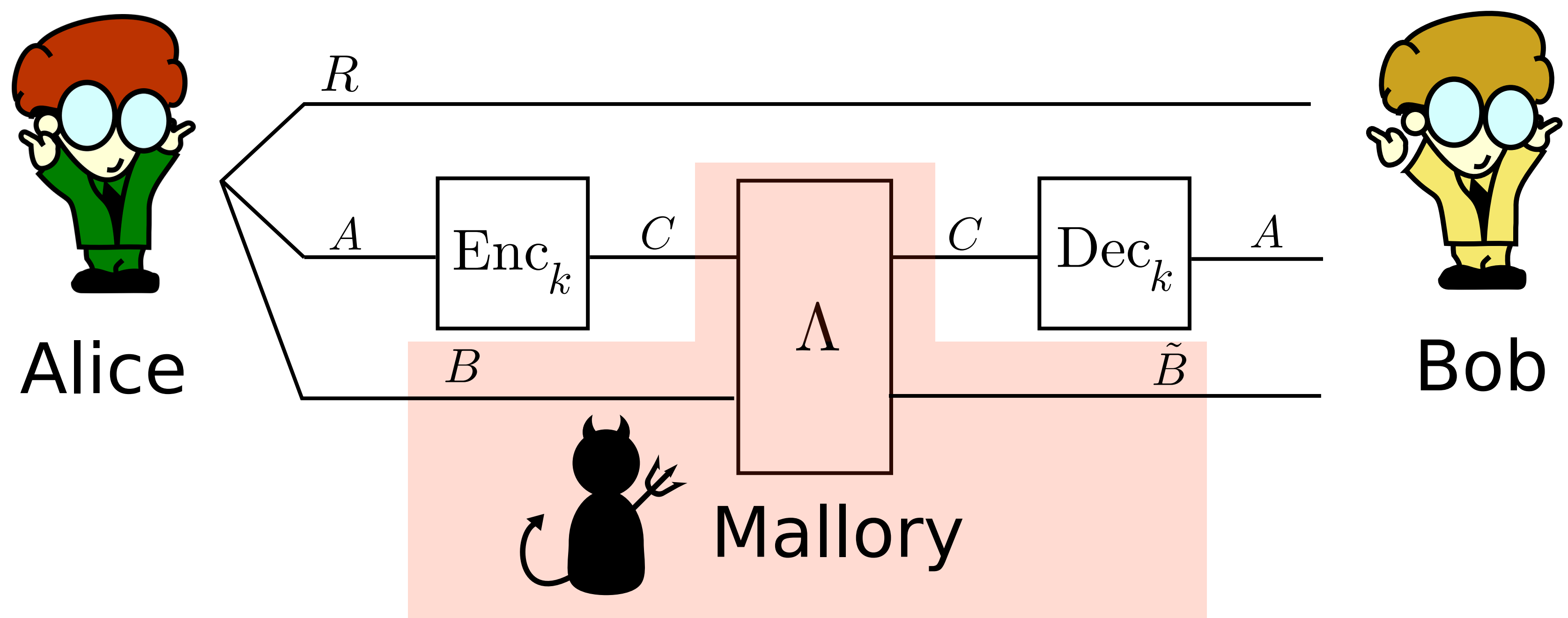

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Full confidentiality 'for free'.

Summary, open questions

Summary:

- We use a technique of Aubrun to give a randomized construction of t-designs in weak norms
- For $t = 2$, our techniques can be used to construct weak designs for the $U \otimes \bar{U}$ and channel twirls
- As an application, we give a randomized construction of a quantum encryption scheme that achieves non-malleability against adversaries without quantum side information with short keys

Open questions:

- For the $U \otimes \bar{U}$ twirl, we only obtain a result for the $1 \to 1$ norm. Can it be strengthened to $d\|\cdot\|_{1\to\infty}$?